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Optimal Control of Parabolic Hemivariational Inequalities

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Abstract. In this paper we study the optimal control of systems driven by parabolic hemivariational inequalities. First, we establish the existence of solutions to a parabolic hemivariational inequality which contains nonlinear evolution operator. Introducing a control variable in the second member and in the multivalued term, we prove the upper semicontinuity property of the solution set of the inequality. Then we use this result and the direct method of the calculus of variations to show the existence of optimal admissible state–control pairs.

Key words: Compact embedding; Control problem; Hemivariational inequality; Monotone operator; Multifunction; Parabolic equation; Selection; Semicontinuity

1. Introduction

In this paper we study the optimal control of systems governed by a parabolic hemivariational inequality of the form

$$\begin{cases} y'(t) + A(t)y(t) + \chi(t) = f(t) + B(t)w(t) \text{ a.e. } t \in (0, T) \\ y(0) = y_0 \\ \chi(x, t) \in \widehat{\beta}(x, t, u(x, t), y(x, t)) \text{ a.e. } (x, t) \in \Omega \times (0, T), \end{cases} \qquad (P)$$

where u and w denote the control variables. This is a nonlinear and nonmonotone boundary value problem. The lower order term $\widehat{\beta}$ is multivalued and discontinuous while the operator $A(t)$ is assumed to be monotone and to satisfy certain coerciveness and boundedness hypotheses.

We note that the problem (P) arises in many important models for distributed parameter control problems and that a large class of identification problems enter our formulation. Let us indicate a problem which is one of the motivations for the study of hemivariational inequality (P). In a subset Ω of $\mathbb{R}^3$, we consider the nonstationary heat conduction equation

$$\frac{\partial y}{\partial t} - \Delta y = f \quad \text{in } \Omega \times (0, T)$$

with the initial condition and suitable boundary ones. Here $y = y(x, t)$ represents the temperature at the point $x \in \Omega$ and time $t \in (0, T)$. It is supposed that $f =$

$\overline{f} + \overline{\overline{f}}$, where $\overline{\overline{f}}$ is given and $\overline{f}$ is a known function of the temperature of the form

$$-\overline{f}(x,t) \in \partial j(x,t,u(x,t),y(x,t)) \quad \text{a.e. } (x,t) \in \Omega \times (0,T).$$

Here $\partial j(x,t,\eta,\xi)$ denotes the generalized gradient of Clarke [7] with respect to the last variable of a function $j\colon \Omega \times (0,T) \times \mathbb{R}^2 \to \mathbb{R}$ which is assumed to be locally Lipschitz in ξ. The multivalued function $\partial j(x,t,\eta,\cdot)\colon \mathbb{R} \to 2^{\mathbb{R}}$ is generally nonmonotone and it includes the vertical jumps. In a physicist's language, it means that the law is characterized by the generalized gradient of a nonsmooth potential j.

The variational formulation of the above problem leads to the hemivariational inequality (P) with $A(t) = -\Delta$, $\widehat{\beta} = \partial j$ and is met, for example, in the nonmonotone nonconvex interior semipermeability problems. For the latter, see e.g. Panagiotopoulos [20], where a temperature control problem in which we regulate the temperature to deviate as little as possible from a given interval, is considered. We remark that the monotone semipermeability problems, leading to variational inequalities, have been studied by Duvaut and Lions in [9] under the assumption that $j(x,t,\eta,\cdot)$ is a proper, lower semicontinuous, convex function which means that $\partial j(x,t,\eta,\cdot)$ is a maximal monotone operator in $\mathbb{R}^2$.

The goal of our study is twofold. First we intend to obtain an existence result for the abstract problem (P). We present a general existence result for weak solutions of problem (P) obtained via Galerkin method combined with a regularization procedure. We note that such a result generalizes the recent one of [16]. A second goal concerns the applications of our abstract result to the control problems which dynamics are described by hemivariational inequality (P) in which the operator $A(t)$ is a nonlinear one. We introduce a control parameter w in the right hand side of the equation and a control variable u in the multivalued relation in (P). Then we investigate the upper semicontinuity of the multifunction $(u,w) \mapsto S(u,w)$, where $S(u,w)$ denotes the set of solutions to hemivariational inequality corresponding to a control (u,w). The existence of optimal pairs is obtained using the direct method of the calculus of variations.

There is a large literature on optimal control of evolution problems. We remark only that the existence and approximation of optimal solutions as well as the necessary optimality conditions for parabolic control problems have been studied, for instance, by Lions [14], Ahmed and Teo [1], Cesari [5] and others for evolution and differential equations and by Barbu [4] and Tiba [26] for variational inequalities. As concerns control problems for hemivariational inequalities, these problems have been treated only recently and so far only in the stationary case by Panagiotopoulos [22], Miettinen [15], Miettinen and Haslinger [17], Haslinger and Panagiotopoulos [11, 12], Denkowski and Migórski [8].

The theorem on existence of solutions for parabolic hemivariational inequality has been delivered by Miettinen in [16] while the finite element approximation for this problem can be found in [18] and in the recent monograph [10].

We mention that the notion of hemivariational inequality was introduced in 1981 by Panagiotopoulos for a description of some nonconvex, nonmonotone, multivalued relations between stress and strain or reaction and displacement appearing in large families of important problems in physics and engineering. The theory of hemivariational inequalities has been proved to be very useful in understanding of many problems of nonsmooth mechanics, such as the debonding of adhesive joints, the delamination of multilayered plates, the ultimate strength of fiber reinforced structures, etc. (see [19], [21], [24]).

An outline of the remainder of the paper is as follows. After preliminaries of Section 2, in Section 3 we state and prove our existence result for (P). We also provide an additional condition on the function β under which we obtain an uniqueness of the solution of the hemivariational inequality. In Section 4 we treat an optimal control problem for (P). Here the dependence of the solution set on the controls is the crutial point in our approach.

2. Notation

In this section we fix the notation and recall some relevant definitions.

Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, $0 < T < +\infty$ and let $Q = \Omega \times (0, T)$. Let V be a real reflexive separable Banach space densely and continuously embedded in the space $H = L^2(\Omega)$. Identifying H with its dual, we have the evolution triple $V \subset H \subset V'$ (see e.g. [13], [27]). We assume that $V \subset H$ compactly. We denote by $\langle \cdot, \cdot \rangle$ the duality of V and V' as well as the inner product on H, by $||\cdot||$, $|\cdot|$ and $||\cdot||_{V'}$ the norms in V, H and V', respectively. Let p and q be constants such that $2 \leqslant p < +\infty$ and $1/p + 1/q = 1$.

For each $r \geqslant 1$, we denote by $L^r(0, T; B)$ the space of strongly measurable B-valued functions $v\colon [0, T] \to B$ such that $\int_0^T ||v(t)||_B^r \, dt < +\infty$. We introduce the spaces $\mathcal{V} = L^p(0, T; V)$, $\mathcal{H} = L^2(0, T; H)$, $\mathcal{V}' = L^q(0, T; V')$ and $\mathcal{W}(V) = \{v \in \mathcal{V} \mid v' \in \mathcal{V}'\}$, where the time derivative is understood in the sense of vector valued distributions. Clearly $\mathcal{W}(V) \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$. The pairing of $\mathcal{V}$ and $\mathcal{V}'$ and the inner product on $\mathcal{H}$ are denoted by $\langle\langle f, v \rangle\rangle = \int_0^T \langle f(t), v(t) \rangle \, dt$. It is well known (cf. e.g. [27]) that $\mathcal{W}(V) \subset C([0, T]; H)$ continuously and $\mathcal{W}(V) \subset \mathcal{H}$ compactly.

Given a Banach space $\mathcal{X}$, the symbol $w - \mathcal{X}$ is always used to indicate the space $\mathcal{X}$ equipped with the weak topology.

A function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is said to be N-measurable (or superpositionally measurable) if for every measurable function $u\colon \Omega \to \mathbb{R}^m$, the function $x \mapsto f(x, u(x))$ is measurable (see Chang [6] for properties of N-measurable functions).

3. An existence result

The goal of this section is to provide conditions under which problem (P) has a solution. To simplify notation we omit the dependence of the right-hand side on the control variable w. Throughout this section the control variable u is fixed in U with $U \subseteq L^2(Q)$.

We consider the following evolution initial boundary value problem: find $y \in \mathcal{W}(V)$ and $\chi \in \mathcal{H}$ such that

$$\begin{cases} y'(t) + A(t)y(t) + \chi(t) = f(t) \quad \text{a.e. } t \in (0,T) \\ y(0) = y_0 \\ \chi(x,t) \in \widehat{\beta}(x,t,u(x,t),y(x,t)) \quad \text{a.e. } (x,t) \in Q. \end{cases} \tag{3.1}$$

We impose the following hypotheses on the data. The first one is standard in the study of evolution equations, cf. [27].

$\underline{H(A)}$: for $t \in (0,T)$ the operator $A(t)\colon V \to V'$ is monotone, hemicontinuous and satisfies the conditions

(i) (coerciveness) there exist constants $c_1 > 0$ and $c_2, c_3 \geqslant 0$ such that for all $v \in V$ and $t \in (0,T)$ we have $\langle A(t)v, v\rangle \geqslant c_1||v||^p - c_2|v|^2 - c_3$.

(ii) (growth condition) there exist a nonnegative function $g \in L^q(0,T)$ and $c_4 > 0$ such that $||A(t)v||_{V'} \leqslant g(t) + c_4||v||^{p-1}$ for all $v \in V$, $t \in (0,T)$.

(iii) (measurability) the function $t \mapsto \langle A(t)z, v\rangle$ is measurable on $(0,T)$ for all $z, v \in V$.

$\underline{H(\beta)}$: $\beta\colon Q \times \mathbb{R}^2 \to \mathbb{R}$, $\beta = \beta(x,t,\eta,\xi)$ is a function satisfying the conditions

(i) β is locally bounded, i.e. $\forall\, r > 0$, $\exists\, c = c(r) > 0$ such that $|\beta(x,t,\eta,\xi)| \leqslant c(r)$ $\forall\, |\eta| \leqslant r$, $\forall\, (x,t) \in Q$ and a.e. $|\xi| \leqslant r$.

(ii) β is continuous in η uniformly with respect to ξ, i.e. $\exists\, \varepsilon_0 > 0$ such that $\forall\, (x,t,\eta,\xi) \in Q \times \mathbb{R}^2$, $\forall\, \delta > 0$, $\exists\, \gamma = \gamma(\delta,x,t,\eta,\xi) > 0$ such that $|\beta(x,t,\eta,\xi') - \beta(x,t,\eta',\xi')| < \delta$ as $|\eta - \eta'| < \gamma$ and $|\xi - \xi'| < \varepsilon_0$.

(iii) $(x,t) \mapsto \beta(x,t,\eta,\xi)$ is continuous on Q for all $\eta \in \mathbb{R}$ and a.e. $\xi \in \mathbb{R}$.

(iv) $(x,t,\xi) \mapsto \beta(x,t,\eta,\xi)$ is measurable in $Q \times \mathbb{R}$ for all $\eta \in \mathbb{R}$.

(v) $|\beta(x,t,\eta,\xi)| \leqslant \alpha(x,t) + c_0(1 + |\eta| + |\xi|)$ $\forall\, (x,t,\eta,\xi) \in Q \times \mathbb{R}^2$ with a nonnegative function $\alpha \in L^2(Q)$ and a positive constant c_0.

$\underline{(H_0)}$: $y_0 \in H$ and $f \in \mathcal{V}'$.

The multivalued function $\widehat{\beta}\colon Q \times \mathbb{R}^2 \to 2^{\mathbb{R}}$ appearing in (3.1) is obtained by "filling in jumps" of a function $\beta(x,t,\eta,\cdot)\colon \mathbb{R} \to \mathbb{R}$ as follows: for $\varepsilon > 0$, $(x,t) \in Q$ and $\eta, \xi \in \mathbb{R}$, let

$$\underline{\beta_\varepsilon}(x,t,\eta,\xi) = \operatorname*{ess\,inf}_{|\tau-\xi|\leqslant\varepsilon} \beta(x,t,\eta,\tau), \quad \overline{\beta_\varepsilon}(x,t,\eta,\xi) = \operatorname*{ess\,sup}_{|\tau-\xi|\leqslant\varepsilon} \beta(x,t,\eta,\tau).$$

For $(x,t) \in Q$ and $\eta, \xi \in \mathbb{R}$ fixed, $\underline{\beta_\varepsilon}$ is an increasing function of ε and $\overline{\beta_\varepsilon}$ is decreasing in ε. Let

$$\underline{\beta}(x,t,\eta,\xi) = \lim_{\varepsilon\to 0} \underline{\beta_\varepsilon}(x,t,\eta,\xi), \quad \overline{\beta}(x,t,\eta,\xi) = \lim_{\varepsilon\to 0} \overline{\beta_\varepsilon}(x,t,\eta,\xi).$$

The multifunction $\widehat{\beta}$ is defined by

$$\widehat{\beta}(x,t,\eta,\xi) = [\underline{\beta}(x,t,\eta,\xi), \overline{\beta}(x,t,\eta,\xi)].$$

The hypothesis $H(\beta)$ has been firstly introduced in [17]. The idea behind it, is to assume conditions as weak as possible for β with respect to the last variable. In order to assure the measurability of the composed function, β has to be smooth in the other variables. The following result may be proved in much the same way as Proposition 1.2 of [17].

PROPOSITION 3.1. *If the function β satisfies hypothesis $H(\beta)(i) \div (iv)$, then the functions $\underline{\beta}$ and $\overline{\beta}$ are N-measurable, that is, the functions*

$$(x,t) \mapsto \underline{\beta}(x,t,u(x,t),y(x,t)) \text{ and } (x,t) \mapsto \overline{\beta}(x,t,u(x,t),y(x,t))$$

are measurable on Q for any $u, y\colon Q \to \mathbb{R}$ measurable.

THEOREM 3.1. *Let $u \in U \subseteq L^2(Q)$ be fixed. If hypotheses $H(A)$, $H(\beta)$ and (H_0) hold, then the problem (3.1) admits at least one solution.*

Proof. Step 1. A priori estimates for solutions of the Galerkin equation.
With the problem (3.1) we associate the approximate one which is obtained as the combination of the regularization of β together with the Galerkin method (cf. [25], [23], [19]). Take $\rho \in C_0^\infty((-1,1))$, $\rho \geqslant 0$, $\int_{-1}^{1} \rho(s)ds = 1$ and let $\rho_n(s) = n\rho(ns)$. Let $\beta_n\colon Q \times \mathbb{R}^2 \to \mathbb{R}$ be a regularization of β given by $\beta_n(x,t,\eta,\xi) = \int_{\mathbb{R}} \beta(x,t,\eta,\xi-\tau)\rho_n(\tau)\,d\tau$. By an argument similar to that of Lemma 1.1 in [17], we deduce that the function β_n (for $1/n < \varepsilon_0/2$) satisfies the Carathéodory conditions (it is measurable in $(x,t) \in Q$ for all $\eta, \xi \in \mathbb{R}$ and it is continuous in $(\eta,\xi) \in \mathbb{R}^2$ for a.e. $(x,t) \in Q$) which implies the N-measurability of β_n.

Let $\{\varphi_1, \varphi_2, \dots\}$ be a basis in V, i.e. $\{\varphi_i\}$ forms an at most countable sequence of elements in V, finitely many $\varphi_1, \dots, \varphi_n$ are linearly independent and $V = cl(\cup_n V_n)$ with $V_n = \operatorname{span}\{\varphi_1, \dots, \varphi_n\}$. Since V is separable the existence of such a basis is guaranteed (cf. Proposition 21.49 in [27]). Moreover, the family $\{V_n\}$ of finite dimensional subspaces of V satisfies

$$\forall\, v \in V \ \ \exists \{v_n\},\ v_n \in V_n \text{ such that } v_n \to v \text{ in } V, \ \text{ as } n \to +\infty. \tag{3.2}$$

Let $\{y_{0n}\}$ be such that $y_{0n} \in V_n$ for $n \in \mathbb{N}$,

$$y_{0n} \to y_0 \text{ in } H, \ \text{ as } n \to +\infty. \tag{3.3}$$

The regularized Galerkin problem for (3.1) is following: find y_n such that $y_n \in L^p(0,T;V_n)$, $y_n' \in L^q(0,T;V_n)$ and

$$\begin{cases} \langle y_n'(t), v_n\rangle + \langle A(t)y_n(t), v_n\rangle + \langle \beta_n(t,u(t),y_n(t)), v_n\rangle = \langle f(t), v_n\rangle \\ \qquad\qquad \text{for a.e. } t \in (0,T) \text{ and all } v_n \in V_n \\ y_n(0) = y_{0n}. \end{cases} \tag{3.4}$$

Let y_n be the solution to (3.4). Using $H(\beta)$(v), we have

$$||\beta_n(\cdot, u(\cdot), y_n(\cdot))||_{\mathcal{H}} \leqslant c_5 \left(1 + ||u||_{\mathcal{H}} + ||y_n||_{\mathcal{H}}\right), \tag{3.5}$$

$$\left| \int_0^t \int_\Omega \beta_n(x, s, u(x, s), y_n(x, s)) y_n(x, s)\, dx\, ds \right| \leqslant c_6 \left(1 + ||u||^2_{L^2(0,t;H)} + ||y_n||^2_{L^2(0,t;H)}\right), \tag{3.6}$$

for $t \in (0, T)$ with positive constants c_5 and c_6. From (3.4), using integration by parts formula, (3.6) and $H(A)(i)$, for $t \in [0, T]$, we have

$$\frac{1}{2}|y_n(t)|^2 - \frac{1}{2}|y_{0n}|^2 = \int_0^t \langle y_n'(s), y_n(s)\rangle\, ds$$

$$= \int_0^t \langle f(s) - \beta_n(s, u(s), y_n(s)) - A(s) y_n(s), y_n(s)\rangle\, ds \leqslant ||f||_{L^q(0,t;V')} ||y_n||_{L^p(0,t;V)} d$$

$$+ c_6 \left(1 + ||u||^2_{L^2(0,t;H)} + ||y_n||^2_{L^2(0,t;H)}\right) - c_1 ||y_n||^p_{L^p(0,t;V)} + c_2 ||y_n||^2_{L^2(0,t;H)} + c_3 T.$$

By virtue of the Cauchy inequality ($ab \leqslant \dfrac{\varepsilon^p}{p}|a|^p + \dfrac{\varepsilon^{-q}}{q}|b|^q$ for $\varepsilon > 0$, $a, b \in \mathbb{R}$) applied (with $\varepsilon = c_1^{1/p}$) to the first term on the right hand side, we obtain

$$|y_n(t)|^2 + 2c_1 \left(1 - \frac{1}{p}\right) ||y_n||^p_{L^p(0,t;V)} \leqslant |y_{0n}|^2 + 2(c_2 + c_6) ||y_n||^2_{L^2(0,t;H)}$$

$$+ 2(c_6 + c_3 T) + \left(\frac{2}{q c_1^{q/p}}\right) ||f||^q_{L^q(0,t;V')} + c_6 ||u||^2_{L^2(0,t;H)}.$$

By Gronwall's inequality and (3.3), it follows that

$$|y_n(t)| \leqslant c_7 \quad \text{for all } t \in [0, T],\ n \in \mathbb{N} \text{ with } c_7 > 0. \tag{3.7}$$

Subsequently, there exists a positive constant c_8 such that

$$||y_n||_{\mathcal{V}} \leqslant c_8 \quad \text{for } n \in \mathbb{N}. \tag{3.8}$$

Thus, it follows from (3.8) and (3.5) that there exists a constant $c_9 > 0$ such that

$$||\beta_n(\cdot, u(\cdot), y_n(\cdot))||_{\mathcal{H}} \leqslant c_9 \quad \text{for } n \in \mathbb{N}. \tag{3.9}$$

Step 2. Existence of a solution to (3.4).
From $H(A)(ii)$, $H(\beta)$(v), (3.7), (3.8) and the fact that $A(t)\colon V \to V'$ is demicontinuous (being monotone and hemicontinuous), we obtain that the function $(t, z) \mapsto \langle A(t)z + \beta_n(\cdot, t, u(\cdot, t), z), \varphi_j\rangle$ satisfies the Carathéodory condition on $[0, T] \times V_n$ and has an integrable majorant.

It follows immediately from the Carathéodory theorem for ordinary differential equation that for every n, the problem (3.4) has a solution $y_n : [0, T] \to V_n$ which is continuous and the derivative y_n' exists for a.e. $t \in [0, T]$. Thus $y_n \in L^p(0, T; V_n)$. Furthermore, by $H(A)(ii)$, $H(\beta)$(v) and (3.8), the functions $t \mapsto \langle A(t)y_n(t), \varphi_j\rangle$, $t \mapsto \langle f(t), \varphi_j\rangle$ and $t \mapsto \langle \beta_n(\cdot, t, u(\cdot, t), y_n(\cdot, t)), \varphi_j\rangle$ belong to $L^q(0, T)$. So, by the regularized Galerkin equation, we deduce that $y_n' \in L^q(0, T; V_n)$.

Step 3. Convergence of subsequences for (3.4).
Setting $(\mathcal{A}v)(t) = A(t)v(t)$ for $v \in \mathcal{V}$ and $t \in (0, T)$, it is easy to see that the operator $\mathcal{A}: \mathcal{V} \to \mathcal{V}'$ is monotone, hemicontinuous, bounded and coercive (see also Chapter 30.3 of [27]). We observe that from (3.4), (3.7), (3.8) and (3.9), we have

$$|\langle\langle y_n', z_n\rangle\rangle| \leqslant ||f||_{\mathcal{V}'}||z_n||_{\mathcal{V}} + \int_0^T \left(g(t) + c_4||y_n(t)||^{p-1}\right)||z_n(t)||\,dt+$$
$$+c_{10}||\beta_n(\cdot, u(\cdot), y_n(\cdot))||_{\mathcal{H}}||z_n||_{\mathcal{V}} \leqslant c_{11}||z_n||_{\mathcal{V}}$$

for $z_n \in L^p(0, T; V_n)$ with $c_{10}, c_{11} > 0$. Hence, by the fact that $\cup_m V_m$ is dense in V, we get

$$||y_n'||_{\mathcal{V}'} \leqslant c_{12} \text{ for } n \in \mathbb{N} \text{ with } c_{12} > 0. \tag{3.10}$$

By the a priori estimates (3.7)–(3.10) and the bound $||\mathcal{A}y_n||_{\mathcal{V}'} \leqslant const(||g||_{L^q(0,T)} + ||y_n||_{\mathcal{V}}^{p/q})$, we conclude that there exists a subsequence, denoted as before, such that

$$y_n \to y \quad \text{weakly in } \mathcal{W}(V), \tag{3.11}$$

$$\beta_n(\cdot, u(\cdot), y_n(\cdot)) \to \chi \quad \text{weakly in } \mathcal{H}, \tag{3.12}$$

$$y_n(T) \to z \quad \text{weakly in } H, \tag{3.13}$$

$$\mathcal{A}y_n \to w \quad \text{weakly in } \mathcal{V}',$$

with some $y \in \mathcal{W}(V)$, $\chi \in \mathcal{H}$, $z \in H$ and $w \in \mathcal{V}'$.

Step 4. The limits y, χ, z and w satisfy $y' + w + \chi = f$, $y(0) = y_0$, $y(T) = z$ and $\mathcal{A}y = w$.
The proof of this step is similar to that of Lemma 30.5 in [27]. We mention only the most important parts. Let $\psi \in C_0^\infty((0, T))$, $v \in V$. Then, by (3.2) there exists $\{v_n\}$, $v_n \in V_n$ such that $v_n \to v$ in V, as $n \to \infty$. Denoting $\Psi(x, t) = \psi(t)v(x)$ and $\Psi_n(x, t) = \psi(t)v_n(x)$, we have $\Psi_n \to \Psi$ in $\mathcal{W}(V)$. From (3.4) after integration by parts we get

$$-\langle\langle y_n, \Psi_n'\rangle\rangle + \langle\langle \mathcal{A}y_n, \Psi_n\rangle\rangle + \langle\langle \beta_n(\cdot, u(\cdot), y_n(\cdot)), \Psi_n\rangle\rangle = \langle\langle f, \Psi_n\rangle\rangle.$$

Letting $n \to \infty$, we obtain

$$-\langle\langle y, \Psi' \rangle\rangle + \langle\langle w, \Psi \rangle\rangle + \langle\langle \chi, \Psi \rangle\rangle = \langle\langle f, \Psi \rangle\rangle$$

and since v and ψ are arbitrary, we have $y' + w + \chi = f$. Next, because the mapping $\mathcal{W}(V) \ni v \mapsto \{v(0), v(T)\} \in H \times H$ is linear and continuous, from (3.3), (3.11) and (3.13), taking weak limits in H, we obtain $z = weak - \lim_n y_n(T) = y(T)$ and $y_0 = weak - \lim_n y_n(0) = y(0)$.

Recall that a monotone and hemicontinuous operator in a reflexive Banach space satisfies condition (M) (cf. [13], [27]), that is, $y_n \to y$ weakly in $\mathcal{V}$, $\mathcal{A}y_n \to w$ weakly in $\mathcal{V}'$ and $\limsup_n \langle\langle \mathcal{A}y_n, y_n \rangle\rangle \leqslant \langle\langle w, y \rangle\rangle$ implies $\mathcal{A}y = w$. To conclude the proof of this step, it is enough to prove that $\limsup_n \langle\langle \mathcal{A}y_n, y_n \rangle\rangle \leqslant \langle\langle w, y \rangle\rangle$. To this end, we only observe that due to (3.11) and (3.12) and the fact that $\mathcal{W}(V) \subset \mathcal{H}$ compactly, we have

$$\lim_n \langle\langle \beta_n(\cdot, u(\cdot), y_n(\cdot)), y_n \rangle\rangle = \langle\langle \chi, y \rangle\rangle.$$

Step 5. Proof that $\chi(x,t) \in \widehat{\beta}(x,t,u(x,t),y(x,t))$ a.e. *in* Q.
We apply Convergence Theorem (see [2], p. 60) to a multivalued mapping $\widehat{\beta}$. Precisely, we observe first that for a.e. $(x,t) \in Q$ and every $\eta \in \mathbb{R}$, $\widehat{\beta}(x,t,\eta,\cdot)\colon \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous since it is closed and locally compact ([2]). Next, from (3.11) and the compactness of embedding $\mathcal{W}(V) \subset \mathcal{H}$, we may suppose that

$$y_n(x,t) \to y(x,t) \quad \text{a.e. } (x,t) \in Q. \tag{3.14}$$

Finally, by (3.14) and the definition of $\widehat{\beta}$ we deduce that for a.e. $(x,t) \in Q$ and for every neighborhood $\mathcal{N}$ of zero in $\mathbb{R}^2$, there exists $N_0 = N_0(x,t,\mathcal{N})$ such that

$$(y_n(x,t), \beta_n(x,t,u(x,t),y_n(x,t))) \in graph\widehat{\beta}(x,t,u(x,t),\cdot) + \mathcal{N}, \quad \forall\, n \geqslant N_0.$$

Having in mind the convergence (3.12), Convergence Theorem ensures that for a.e. $(x,t) \in Q$ we have $\chi(x,t) \in \overline{co}\,\widehat{\beta}(x,t,u(x,t),y(x,t)) = \widehat{\beta}(x,t,u(x,t),y(x,t))$, which completes the proof of the theorem. □

As an example of the operator which satisfies the hypothesis $H(A)$ we consider the following one.

EXAMPLE 3.1. Let us fix three positive real constants m_0, m_1, m_2 and $0 < \alpha \leqslant 1$. By $S = S(m_0, m_1, m_2, \alpha)$ we denote the class of functions $a\colon Q \times \mathbb{R}^N \to \mathbb{R}^N$ which satisfy the following conditions:

(i) $|a(x,t,0)| \leqslant m_0$, a.e. in Q,
(ii) $a(\cdot,\cdot,\xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^N$,
(iii) $|a(x,t,\xi) - a(x,t,\eta)| \leqslant m_1(1 + |\xi| + |\eta|)^{p-1-\alpha}|\xi - \eta|^{\alpha}$, a.e. in Q and for every $\xi, \eta \in \mathbb{R}^N$,

(iv) $(a(x,t,\xi) - a(x,t,\eta), \xi - \eta) \geqslant m_2|\xi - \eta|^p$, a.e. in Q and for every $\xi, \eta \in \mathbb{R}^N$.

We consider a space V which is a closed subspace of $W^{1,p}(\Omega)$ such that $W_0^{1,p}(\Omega) \subseteq V$. Let $A(t)\colon\ W^{1,p}(\Omega) \to V'$ be the operator defined by

$$\langle A(t)y, v\rangle = \int_\Omega (a(x,t,Dy), Dv)\ dx$$

with a function $a \in S(m_0, m_1, m_2, \alpha)$, for $y \in W^{1,p}(\Omega)$, $v \in V$ and $t \in (0,T)$. We also define $L(t)\colon W^{1,p}(\Omega) \to W^{-1,q}(\Omega)$ by setting

$$\langle L(t)y, v\rangle \equiv \langle A(t)y, v\rangle \quad \text{for } \ y \in W^{1,p}(\Omega),\ v \in W_0^{1,p}(\Omega),$$

that is $L(t)y = -\operatorname{div} a(x,t,Dy)$. Then for each fixed $t \in (0,T)$, the operator $L(t)$ satisfies the hypothesis $H(A)$ and the operator $A(t)\colon V \to V'$ satisfies the hypothesis $H(A)$ with $H_0^1(\Omega) \subseteq V \subseteq H^1(\Omega)$.

The model example of the function in the class S is following $a(x,t,\xi) = d(x,t)|\xi|^{p-2}\xi$, where $2 \leqslant p < +\infty$ and $d\colon Q \to \mathbb{R}$ is a measurable function satisfying $0 < c_1 \leqslant d(x,t) \leqslant c_2$ for all $(x,t) \in Q$. It is easy to check that $a \in S(m_0, c_2(p-1), c_1 2^{2-p}, 1)$ for every $m_0 > 0$. The corresponding operator reduces in this case to the operator $-\operatorname{div}\big(d(x,t)|Dy|^{p-2}Dy\big)$, which for $d(x,t) \equiv 1$ coincides with the p-Laplacian one.

The next result provides a certain growth condition on the function β under which the solution of the hemivariational inequality is unique.

PROPOSITION 3.2. *Let the operator $A(t)\colon V \to V'$ be monotone and satisfy $H(A)(iii)$. Let $\beta\colon Q \times \mathbb{R}^2 \to \mathbb{R}$ satisfy $H(\beta)(i) \div (iv)$ and the following condition*

$$\operatorname*{ess\,inf}_{\xi_1 \neq \xi_2} \frac{\beta(x,t,\eta,\xi_1) - \beta(x,t,\eta,\xi_2)}{\xi_1 - \xi_2} \geqslant -k \quad \text{for all } (x,t,\eta) \in Q \times \mathbb{R} \tag{3.15}$$

with $k > 0$. Then the problem (3.1) has at most one solution.

Proof. Let (y_1, χ_1), (y_2, χ_2) be solutions to (3.1). We will show that $(y_1, \chi_1) = (y_2, \chi_2)$. For $i = 1, 2$ we have

$$y_i'(t) + A(t)y_i(t) + \chi_i(t) = f(t) \quad \text{a.e. } t \in (0,T) \tag{3.16}$$

$$y_i(0) = y_0 \tag{3.17}$$

$$\chi_i(x,t) \in \widehat{\beta}(x,t,u(x,t), y_i(x,t)) \quad \text{a.e. } (x,t) \in Q. \tag{3.18}$$

Subtracting the two equations in (3.16), multiplying the result by $y_1(t) - y_2(t)$, integrating by parts and using (3.17), we obtain

$$0 = \frac{1}{2}|y_1(t) - y_2(t)|^2 - \frac{1}{2}|y_1(0) - y_2(0)|^2$$

$$+\int_0^t \langle A(s)y_1(s) - A(s)y_2(s), y_1(s) - y_2(s)\rangle\, ds + \int_0^t \langle \chi_1(s) - \chi_2(s), y_1(s) - y_2(s)\rangle\, ds$$

$$\geqslant \frac{1}{2}|y_1(t) - y_2(t)|^2 + \int_0^t \langle \chi_1(s) - \chi_2(s), y_1(s) - y_2(s)\rangle\, ds \tag{3.19}$$

for $t \in [0, T]$. From (3.15), we get

$$\inf_{\xi_1 > \xi_2} \frac{\underline{\beta}(x, t, \eta, \xi_1) - \overline{\beta}(x, t, \eta, \xi_2)}{\xi_1 - \xi_2} \geqslant -k \quad \text{for all } (x, t, \eta) \in Q \times \mathbb{R}.$$

Let $\omega_1(s) = \{x \in \Omega : y_1(x, s) > y_2(x, s)\}$ and $\omega_2(s) = \{x \in \Omega : y_2(x, s) > y_1(x, s)\}$. Using the last inequality, the definition of $\widehat{\beta}$ and (3.18), we have

$$\begin{aligned}
&\langle \chi_1(s) - \chi_2(s), y_1(s) - y_2(s)\rangle \\
&\quad = \int_{\omega_1(s)} (\chi_1(x, s) - \chi_2(x, s))\, (y_1(x, s) - y_2(x, s))\, dx \\
&\qquad + \int_{\omega_2(s)} (\chi_1(x, s) - \chi_2(x, s))\, (y_1(x, s) - y_2(x, s))\, dx \\
&\quad \geqslant \int_{\omega_1(s)} \Big[\underline{\beta}(x, s, u(x, s), y_1(x, s)) - \overline{\beta}(x, s, u(x, s), y_2(x, s))\Big] \\
&\qquad (y_1(x, s) - y_2(x, s))\, dx \\
&\quad + \int_{\omega_2(s)} \Big[\underline{\beta}(x, s, u(x, s), y_2(x, s)) - \overline{\beta}(x, s, u(x, s), y_1(x, s))\Big] \\
&\qquad (y_2(x, s) - y_1(x, s))\, dx \\
&\quad \geqslant -k \int_{\omega_1(s)} (y_1(x, s) - y_2(x, s))^2\, dx - k \int_{\omega_2(s)} (y_2(x, s) \\
&\qquad - y_1(x, s))^2\, dx = -k|y_1(s) - y_2(s)|^2.
\end{aligned}$$

From (3.19) and the last expression we have

$$\frac{1}{2}|y_1(t) - y_2(t)|^2 \leqslant k \int_0^t |y_1(s) - y_2(s)|^2\, ds \quad \text{for } t \in [0, T].$$

Invoking the Gronwall inequality, we easily get $y_1 = y_2$. Finally from (3.16) we deduce $\chi_1 = \chi_2$ which completes the proof. □

COROLLARY 3.1. *Under the hypotheses of Theorem 3.1 and the assumption (3.15), the problem (3.1) possesses a unique solution.*

4. An optimal control problem

In this section we shall study an optimal control problem for a system which dynamics is described by the parabolic hemivariational inequality (3.1) with a nonlinear operator $A(t)$ on a Sobolev space.

First we study the dependence of the solution set of (3.1) on the control. We consider a space V satisfying

$$V \text{ is a closed subspace of } W^{1,p}(\Omega) \text{ such that } W_0^{1,p}(\Omega) \subseteq V. \tag{4.20}$$

The case $V = W_0^{1,p}(\Omega)$ corresponds to Dirichlet boundary condition and the case $V = W^{1,p}(\Omega)$ corresponds to Neumann boundary data. Intermediate cases correspond to mixed type boundary conditions.

Let $U_{ad} \times W_{ad}$ be a nonempty subset (representing the set of admissible controls) of $L^2(Q) \times L^2(0, T; W)$ (the space of controls), where W is a Banach space.

For any given $(u, w) \in U_{ad} \times W_{ad}$, we denote by $S(u, w)$ the set of solutions in $\mathcal{W}(V)$ of the problem:

$$\begin{cases} y'(t) + A(t)y(t) + \chi(t) = f(t) + B(t)w(t) \quad \text{a.e. } t \in (0, T) \\ y(0) = y_0 \\ \chi(x, t) \in \widehat{\beta}(x, t, u(x, t), y(x, t)) \quad \text{a.e. } (x, t) \in Q. \end{cases} \tag{4.21}$$

The optimal control problem for (4.21) is formulated as follows

$$\min_{(u,w)\in U_{ad}\times W_{ad}} \ \min_{y\in S(u,w)} J(y, u, w), \tag{CP}$$

where $J\colon \mathcal{W}(V) \times L^2(Q) \times L^2(0, T; W) \to \overline{\mathbb{R}}$ is a given cost functional.

We present a result on the upper semicontinuity of the solution set $S(u, w)$ of (4.21).

Hypotheses:

$\underline{H(U)}$: U_{ad} is a nonempty, compact subset of $L^2(Q)$, W_{ad} is a nonempty, closed convex subset of $L^2(0, T; W)$, where W is a reflexive separable Banach space.

$\underline{H(B)}$: $B \in L^q(0, T; \mathcal{L}(W, H))$, where $\mathcal{L}(X, Y)$ denotes the space of linear continuous operators from X into Y.

PROPOSITION 4.1. *Let V be a space satisfying (4.20). Assume that the hypotheses $H(U)$, $H(B)$, $H(\beta)$, $H(A)$ and (H_0) hold. Then the multivalued mapping*

$$L^2(Q) \times L^2(0, T; W) \supset U_{ad} \times W_{ad} \ni (u, w) \mapsto S(u, w) \in 2^{\mathcal{W}(V)}$$

has a closed graph in the $L^2(Q) \times (w - L^2(0, T; W)) \times (w - \mathcal{W}(V))$ topology.

Proof. Let $(u_m, w_m) \in U_{ad} \times W_{ad}$,

$$u_m \to u \text{ in } L^2(Q) \text{ and a.e. in } Q, \tag{4.22}$$

$w_m \to w$ weakly in $L^2(0, T; W)$, as $m \to \infty$, $y_m \in S(u_m, w_m)$ and

$$y_m \to y \quad \text{weakly in } \mathcal{W}(V), \text{ as } m \to \infty. \tag{4.23}$$

Thus for every $m \in \mathbb{N}$, we have

$$\begin{aligned} &y_m'(t) + A(t)y_m(t) + \chi_m(t) = f(t) + B(t)w_m(t) \quad \text{a.e. } t \in (0, T) \\ &y_m(0) = y_0 \end{aligned} \tag{4.24}$$

$$\chi_m(x, t) \in \widehat{\beta}(x, t, u_m(x, t), y_m(x, t)) \quad \text{a.e. in } Q. \tag{4.25}$$

We shall show that $y \in S(u, w)$. First of all, by the hypothesis $H(U)$, we may assume that $u \in U_{ad}$. We may also notice that $w \in W_{ad}$ since W_{ad} is weakly closed by $H(U)$ and Mazur's theorem. Since $\mathcal{W}(V) \subset \mathcal{H}$ is compact, from (4.23), we have

$$y_m \to y \quad \text{in } \mathcal{H} \text{ and a.e. in } Q \tag{4.26}$$

for a next subsequence. Due to the growth condition $H(\beta)$(v), (4.22) and (4.26), we get

$$||\chi_m||_{\mathcal{H}} \leqslant c_5\,(1 + ||u_m||_{\mathcal{H}} + ||y_m||_{\mathcal{H}}) \leqslant const.,$$

and so we have

$$\chi_m \to \chi \quad \text{weakly in } \mathcal{H} \text{ with some } \chi \in \mathcal{H}. \tag{4.27}$$

From (4.22), (4.25), (4.26), (4.27) and the fact that $\widehat{\beta}$ is convex, compact valued multifunction, we obtain by using Convergence Theorem of [2] that

$$\chi(x, t) \in \widehat{\beta}(x, t, u(x, t), y(x, t)) \quad \text{a.e. in } Q. \tag{4.28}$$

Moreover, by hypothesis $H(B)$, we have

$$\mathcal{B}w_m - \chi_m \to \mathcal{B}w - \chi \quad \text{weakly in } \mathcal{H}, \tag{4.29}$$

where $(\mathcal{B}w)(t) = B(t)w(t)$. Since the set $\{y \in \mathcal{W}(V) : y(0) = y_0\}$ (being closed and convex) is weakly closed (by Mazur's theorem again), from (4.23), we immediately get

$$y(0) = y_0. \tag{4.30}$$

To finish the proof it is enough to show that

$$\mathcal{A}y_m \to \mathcal{A}y \quad \text{weakly in } \mathcal{V}'. \tag{4.31}$$

Analogously to *Step 4* in the proof of Theorem 3.1 the operator $\mathcal{A}$ satisfies conditions:

$$\mathcal{A}y_m \to z \text{ weakly in } \mathcal{V}' \text{ for } y_m \to y \text{ weakly in } \mathcal{V} \text{ and}$$
$$\limsup \langle\langle \mathcal{A}y_m, y_m \rangle\rangle \leqslant \langle\langle z, y \rangle\rangle$$

Since $\mathcal{A}$ is of type (M), it follows that $\mathcal{A}y = z$, so the convergence (4.31) is proved.

Finally, using (4.23), (4.29), (4.31) and letting $m \to \infty$, from (4.24), we conclude that

$$y'(t) + (\mathcal{A}y)(t) + \chi(t) = f(t) + (\mathcal{B}w)(t) \text{ for a.e. } t \in (0, T).$$

This together with (4.28) and (4.30) gives $y \in S(u, w)$, which completes the proof of the proposition. □

REMARK 4.1. The Proposition 4.1 still holds if in the hypothesis $H(U)$ we assume only that U_{ad} is a closed subset of $L^2(Q)$.

Now we are in a position to give a result on existence of optimal solutions to (CP). We restrict ourselves to the case $p = 2$. We admit

$\underline{H(J)}$: $J : \mathcal{H} \times \mathcal{H} \times L^2(0, T; W) \to \mathbb{R}$ is a cost functional of the form

$$J(y, u, w) = \int_0^T L\,(t, y(t), u(t), w(t))\; dt,$$

where $L : [0, T] \times H \times H \times W \to \mathbb{R} \cup \{+\infty\}$ is a measurable function which satisfies three conditions:

(i) $L\,(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $H \times H \times W$, a.e. $t \in (0, T)$.
(ii) $L\,(t, y, \cdot, \cdot)$ is convex on $H \times W$, for all $y \in H$ and a.e. t.
(iii) there exist $M > 0$ and $\psi \in L^1(0, T)$ such that for all $y \in H$, $u \in H$, $w \in W$ and a.e. t, we have $L\,(t, y, u, w) \geqslant \psi(t) + M\left(|y| + |u|^2 + ||w||_W^2\right)$.

THEOREM 4.1. *Let $H_0^1(\Omega) \subseteq V \subseteq H^1(\Omega)$. If the hypotheses $H(U)$, $H(B)$, $H(\beta)$, $H(A)$ and $H(J)$ hold, then the control problem (CP) has a solution.*

Proof. By Theorem 3.1, we have that $S(u, w) \neq \emptyset$ for all $(u, w) \in U_{ad} \times W_{ad}$. Let $(y_k, u_k, w_k) \in S(u_k, w_k) \times U_{ad} \times W_{ad}$ be a minimizing sequence, i.e. $\lim_k J(y_k, u_k, w_k) = \inf\{J(y, u, w) : y \in S(u, w),\ (u, w) \in U_{ad} \times W_{ad}\}$. From the hypothesis $H(U)$ and the fact that $\{w_k\}$ is bounded in W_{ad} (by the coercivity of J), by passing to a subsequence if necessary, we may assume that

$$(u_k, w_k) \to (u^*, w^*) \quad \text{in } L^2(Q) \times (w - L^2(0, T; W)) \tag{4.32}$$

with $u^* \in U_{ad}$. As W_{ad} is weakly closed, we get $w^* \in W_{ad}$.

We shall derive some a priori bounds for solutions $y_k \in S(u_k, w_k)$. For this purpose we use a similar argument as in Section 3 and replace (4.21) by the equivalent form

$$\begin{cases} \langle y_k'(t), v\rangle + \langle A(t)y_k(t), v\rangle + \langle \chi_k(t), v\rangle = \langle f(t), v\rangle + \langle B(t)w_k(t), v\rangle \\ \qquad\qquad\qquad\qquad \text{for all } v \in V \text{ and a.e. } t \in (0, T), \\ y_k(0) = y_0 \\ \chi_k(x,t) \in \widehat{\beta}(x, t, u_k(x,t), y_k(x,t)) \quad \text{a.e. } (x,t) \in Q. \end{cases} \tag{4.33}$$

Using $H(\beta)$(v), we have $||\beta(\cdot, u_k(\cdot), y_k(\cdot))||_{\mathcal{H}} \leqslant c_5\,(1 + ||u_k||_{\mathcal{H}} + ||y_k||_{\mathcal{H}})$, and consequently,

$$||\chi_k||_{\mathcal{H}} \leqslant c_5\,(1 + ||u_k||_{\mathcal{H}} + ||y_k||_{\mathcal{H}}), \tag{4.34}$$

$$\left|\int_0^t \int_\Omega \chi_k(x,s) y_k(x,s)\,dx\,ds\right| \leqslant c_6 \left(1 + ||u_k||^2_{L^2(0,t;H)} + ||y_k||^2_{L^2(0,t;H)}\right), \tag{4.35}$$

for $t \in (0, T)$ with positive constants c_5 and c_6.

From (4.33), using integration by parts formula, (4.35), $H(A)$(i) and the Cauchy inequality, for $t \in [0, T]$, we obtain

$$\frac{1}{2}\left(|y_k(t)|^2 - |y_k(0)|^2\right) \leqslant \frac{1}{2c_1}||f||^2_{\mathcal{V}'} + \frac{c_1}{2}||y_k||^2_{L^2(0,t;V)} + \frac{1}{2}||\mathcal{B}w_k||^2_{\mathcal{H}} + \frac{1}{2}||y_k||^2_{L^2(0,t;H)}$$

$$+ c_6\left(1 + ||u_k||^2_{\mathcal{H}} + ||y_k||^2_{L^2(0,t;H)}\right) - c_1||y_k||^2_{L^2(0,t;V)} + c_2||y_k||^2_{L^2(0,t;H)} + c_3 T.$$

Hence

$$\frac{1}{2}|y_k(t)|^2 + \frac{c_1}{2}||y_k||^2_{L^2(0,t;V)} \leqslant \frac{1}{2}|y_0|^2 + \frac{1}{2c_1}||f||^2_{\mathcal{V}'} + \frac{1}{2}||B||^2_{L^2(0,T;\mathcal{L}(W,H))}||w_k||^2_{L^2(0,T;W)}$$

$$+ c_6 + c_3 T + c_6||u_k||^2_{\mathcal{H}} + \left(\frac{1}{2} + c_6 + c_2\right)||y_k||^2_{L^2(0,t;H)}.$$

By Gronwall's inequality, it follows that $|y_k(t)| \leqslant c_{13}$ for all $t \in [0, T]$, and subsequently,

$$||y_k||_{\mathcal{V}} \leqslant c_{14} \quad \text{for } k \in \mathbb{N} \text{ with } c_{14} > 0. \tag{4.36}$$

We observe that from (4.33), $H(A)$(ii) and Hölder's inequality, we have

$$|\langle\langle y_k', v\rangle\rangle| \leqslant ||f||_{\mathcal{V}'}||v||_{\mathcal{V}} + ||\mathcal{B}w_k||_{\mathcal{H}}||v||_{\mathcal{H}} + ||\chi_k||_{\mathcal{H}}||v||_{\mathcal{H}} \\ +2\left(||g||_{L^2(0,T)} + c_4||y_k||_{\mathcal{V}}\right)||v||_{\mathcal{V}}$$

for $v \in \mathcal{V}$. Hence, by (4.32), (4.34) and (4.36) we get

$$||y_k||_{\mathcal{V}'} \leqslant c_{15} \quad \text{for } k \in \mathbb{N} \text{ with } c_{15} > 0.$$

From this and (4.36) we conclude that $\{y_k\}$ is bounded in $\mathcal{W}(V)$. Hence, by passing to a subsequence if necessary, we assume that

$$y_k \to y^* \quad \text{weakly in } \mathcal{W}(V). \tag{4.37}$$

Then invoking Proposition 4.1 and the convergences (4.32), (4.37), we get $y^* \in S(u^*, w^*)$, i.e. the triplet (y^*, u^*, w^*) is admissible for (CP).

Finally, from hypothesis $H(J)$(iii), we deduce that

$$L(t, y, u, w) \geqslant \psi(t) - M\left(|y| + |u| + ||w||_W\right), \quad \forall\, y \in H,\ u \in U,\ w \in W, \text{ a.e. } t.$$

Now applying Theorem 2.1 from [3], we obtain J is sequentially lower semicontinuous on $L^1(0, T; H) \times (w - L^1(0, T; H \times W))$ and, in consequence, J is sequentially lower semicontinuous on $\mathcal{H} \times (w - L^2(0, T; H \times W))$. Hence and from (4.32) and (4.37), we conclude that $J(y^*, u^*, w^*) = \inf\{J(y, u, w) : y \in S(u, w),\ (u, w) \in U_{ad} \times W_{ad}\}$, which completes the proof of the theorem. □

REMARK 4.2. The hypothesis $H(J)$ incorporates the quadratic cost functionals considered by Lions [14]. In particular, we may take $L(t, y, u, w) = |Cy - \overline{y}|^2 + (N_1 u, u)_H + (N_2 w, w)_W$, where $C, N_1 \in \mathcal{L}(H)$, $N_2 \in \mathcal{L}(W)$, W being a Hilbert space, $(N_i z, z) \geqslant \nu_i |z|^2$ with $\nu_i > 0$, $i = 1, 2$ and $\overline{y}$ is a given element (desired output) in H. Similarily, we can consider a terminal cost functional as well as a combination of the two. For details, we refer to [14] and [1].

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